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DEGENERATE COMPLETE BELL POLYNOMIALS AND NUMBERS

TAEKYUN KIM

Abstract. Recently, several authors have studied complete Bell polynomials and numbers. In this paper, we consider degenerate complete Bell polynomials and numbers and we give some explicit formulas for these numbers and polynomials related to degenerate Stirling numbers.

1. Introduction

As is well known, the Bell polynomials (also called Tochard polynomials or exponential polynomials) are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [6, 9])}. \tag{1.1}$$

In [6], the degenerate Bell polynomials are given by the generating function to be

$$e^{x((1+\lambda t)^{\frac{1}{\lambda}}-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.2}$$

Note that $\lim_{\lambda\to 0} Bel_{n,\lambda}(x) = Bel_n(x)$, $(n \geq 0)$. When $x = 1$, $Bel_{n,\lambda} = Bel_{n,\lambda}(1)$ are called degenerate Bell numbers. For $n \geq 0$, the Stirling number of the first kind is defined by the generating function to be

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \ (k \geq 0), \quad \text{(see [1-12])}. \tag{1.3}$$

and the Stirling number of the second kind is given by the generating function to be

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$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\ (k\geq 0). \tag{1.4}$$

For $\lambda\in\mathbb{R}$, the degenerate Stirling number of the second kind is introduced by the generating function to be

$$\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\ (k\geq 0),\quad \text{(see [5, 7])}. \tag{1.5}$$

By (1.5), we easily get

$$\begin{aligned}
&\frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-1}(1+\lambda t)^{\frac{l}{\lambda}}\\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{\infty}\binom{\frac{l}{\lambda}}{m}\lambda^m t^m=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}\left(\frac{l}{\lambda}\right)_n\lambda^n\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}\right)\frac{t^n}{n!},
\end{aligned} \tag{1.6}$$

where $(x)_0=1$, $(x)_n=x(x-1)\cdots(x-n+1)$, $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n\geq 1)$. Thus, by (1.5) and (1.6), we get

$$\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(l)_{n,\lambda}=\begin{cases}0 & \text{if } n<k\\ S_{2,\lambda}(n,k) & \text{if } n\geq k.\end{cases} \tag{1.7}$$

From (1.2), we note that

$$Bel_{n,\lambda}(x)=e^{-x}\sum_{k=0}^{\infty}\frac{1}{k!}(k)_{n,\lambda}x^k,\ (n\geq 0), \tag{1.8}$$

and

$$Bel_{n,\lambda}=\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}(k)_{n,\lambda},\quad \text{(see [6, 7])}. \tag{1.9}$$

The exponential partial Bell polynomials are the polynomials which are given by the generating function to be

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x^m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \cdots, x_{n-k+1})\frac{t^n}{n!}, \ (k \geq 0), \quad (\text{see [9]}). \tag{1.10}$$

Note that

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{i_1! i_2! \cdots i_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{i_1}\left(\frac{x_2}{2!}\right)^{i_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}},$$

where the summation is over all integer $i_1, i_2 \cdots, i_{n-k+1} \geq 0$ such that $i_1 + i_2 + \cdots + i_{n-k+1} = k$ and $i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n$, (see [1]). The (exponential) complete Bell polynomials are defined by

$$\exp\left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right) = \sum_{n=0}^{\infty} Bel_n(x_1, x_2, \cdots, x_n)\frac{t^n}{n!}, \quad (\text{see [2, 12]}). \tag{1.11}$$

Thus, by (1.10) and (1.11), we get

$$Bel_n(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \cdots, x_n), \ (n \geq 0). \tag{1.12}$$

In this paper, we consider degenerate complete Bell polynomials and numbers and we give some new explicit formulas for these numbers and polynomials.

2. Degenerate complete Bell polynomials and numbers

Let us consider the following exponential incomplete degenerate Bell polynomials as follows:

$$\begin{aligned} &B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda}, \cdots, x_{n-k+1}(1)_{n-k+1,\lambda}) \\ &= \sum \frac{n!}{i_1! i_2! \cdots i_{n-k+1}!}\left(\frac{(1)_{1,\lambda}x_1}{1!}\right)^{j_1}\left(\frac{(1)_{2,\lambda}x_2}{2!}\right)^{j_2}\cdots\left(\frac{(1)_{n-k+1,\lambda}x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \end{aligned} \tag{2.1}$$

where $j_1 + j_2 + \cdots + j_{n-k+1} = k$, $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$. Now, we define the degenerate complete Bell polynomials which are given by

$$\begin{aligned} &Bel_n^{(\lambda)}(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda} \cdots, x_{n-k+1}(1)_{n-k+1,\lambda}), \\ &Bel_0^{(\lambda)}(x_1, x_2, \cdots, x_n) = 0, \ \lambda \in \mathbb{R} \text{ and } n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

The exponential incomplete Bell polynomials are also given by the double series of generating function to be

$$\begin{aligned}\exp\left(u\sum_{j=1}^{\infty}x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)&=\sum_{k=0}^{\infty}u^k\frac{1}{k!}\left(\sum_{j=1}^{\infty}x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^k\\&=1+\sum_{k=1}^{\infty}u^k\frac{1}{k!}\left(\sum_{j=1}^{\infty}x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^k\\&=1+\sum_{k=1}^{\infty}u^k\sum_{n=k}^{\infty}B_{n,k}(x_1(1)_{1,\lambda},\cdots,x_{n-k+1}(1)_{n-k+1,\lambda})\frac{t^n}{n!}\\&=1+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}u^kB_{n,k}(x_1(1)_{1,\lambda},\cdots,x_{n-k+1}(1)_{n-k+1,\lambda})\right)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

Now, we define the extended degenerate complete Bell polynomials which are given by the generating function to be

$$\exp\left(u\sum_{j=1}^{\infty}x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)=\sum_{n=0}^{\infty}Bel_n^{(\lambda)}(u|x_1,x_2,\cdots,x_n)\frac{t^n}{n!}.\tag{2.4}$$

Thus, by (2.3) and (2.4), we get

$$Bel_n^{(\lambda)}(u|x_1,x_2,\cdots,x_n)=\sum_{k=0}^{n}u^kB_{n,k}(x_1(1)_{1,\lambda},\cdots,x_{n-k+1}(1)_{n-k+1,\lambda})\tag{2.5}$$

By (2.5), we easily get

$$Bel_n^{(\lambda)}(1|x_1,x_2,\cdots,x_n)=Bel_n^{(\lambda)}(x_1,x_2,\cdots,x_n),\ Bel_0^{(\lambda)}(x_1,x_2,\cdots,x_n)=1.\tag{2.6}$$

Note that $\lim_{\lambda\to 0} Bel_n^{(\lambda)}(1|x_1,x_2,\cdots,x_n) = Bel_n(x_1,\cdots,x_n)$, $(n \geq 0)$. Now, we observe that

$$\begin{aligned}
&\exp\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right) = 1+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^n \frac{1}{n!}\\
&= 1+\frac{1}{1!}\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}+\frac{1}{2!}\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^2+\frac{1}{3!}\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^3+\cdots\\
&= 1+\frac{x_1}{1!}(1)_{1,\lambda}t+\left(\frac{x_2}{2!}(1)_{2,\lambda}+\frac{x_1^2}{2!}(1)_{1,\lambda}^2\right)t^2\\
&\quad+\left(\frac{x_3}{3!}(1)_{3,\lambda}+\frac{x_1x_2}{2!}(1)_{1,\lambda}(1)_{2,\lambda}+\frac{x_1^3}{3!}(1)_{1,\lambda}\right)t^3+\cdots\\
&= \sum_{n=0}^{\infty}\left(\sum_{m_1+2m_2+3m_3+\cdots+nm_n=n}\frac{n!}{m_1!m_2!\cdots m_n!}\left(\frac{x_1(1)_{1,\lambda}}{1!}\right)^{m_1}\left(\frac{x_2(1)_{2,\lambda}}{2!}\right)^{m_2}\cdots\right.\\
&\quad\left.\times\left(\frac{x_n(1)_{n,\lambda}}{n!}\right)^{m_n}\right)\frac{t^n}{n!}
\end{aligned} \tag{2.7}$$

From (2.3), we note that

$$\begin{aligned}
\exp\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)^k\\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} B_{n,k}(x_1(1)_{1,\lambda},x_2(1)_{2,\lambda},\cdots,x_{n-k+1}(1)_{n-k+1,\lambda})\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} B_{n,k}(x_1(1)_{1,\lambda},x_2(1)_{2,\lambda},\cdots,x_{n-k+1}(1)_{n-k+1,\lambda})\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty} Bel_n^{(\lambda)}(x_1,x_2\cdots x_n)\frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

Comparing the coefficients on the both sides of (2.7) and (2.8), we have

$$\begin{aligned}
&Bel_n^{(\lambda)}(x_1, x_2, \cdots, x_n) \\
&= \sum_{m_1+2m_2+3m_3+\cdots+nm_n=n} \frac{n!}{m_1!m_2!\cdots m_n!} \left(\frac{x_1(1)_{1,\lambda}}{1!}\right)^{m_1} \cdots \left(\frac{x_n(1)_{n,\lambda}}{n!}\right)^{m_n},
\end{aligned} \tag{2.9}$$

where $n \geq 0$.

From (2.8), we note that

$$\begin{aligned}
\exp\left(x\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right) &= 1 + \sum_{k=1}^{\infty} x^k \frac{1}{k!}\left(\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right) \\
&= 1 + \sum_{k=1}^{\infty} x^k \sum_{n=k}^{\infty} B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \cdots (1)_{n-k+1,\lambda})\frac{t^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} x^k B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \cdots (1)_{n-k+1,\lambda})\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \cdots (1)_{n-k+1,\lambda})\right)\frac{t^n}{n!}
\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}
\exp\left(x\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right) &= \exp\left(x\sum_{j=1}^{\infty}\lambda^j\left(\frac{1}{\lambda}\right)_j\frac{t^j}{j!}\right) \\
&= \exp\left(x\left(\sum_{j=0}^{\infty}\binom{\frac{1}{\lambda}}{j}\lambda^j t^j - 1\right)\right) \\
&= \exp(x((1+\lambda t)^{\frac{1}{\lambda}} - 1)) \\
&= \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we get

$$\sum_{k=0}^{n} x^k B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \cdots, (1)_{n-k+1,\lambda}) = Bel_{n,\lambda}(x), \ (n \geq 0). \tag{2.12}$$

From (2.8), we can easily derive the following equation.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} Bel_n^{(\lambda)}(x,x,\cdots,x)\frac{t^n}{n!} = \exp\left(x\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right)\\
&= \sum_{k=0}^{\infty} x^k \frac{1}{k!}\left(\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right)^k = \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} B_{n,k}((1)_{1,\lambda},(1)_{2,\lambda},\cdots,(1)_{n-k+1,\lambda})\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k B_{n,k}((1)_{1,\lambda},(1)_{2,\lambda},\cdots,(1)_{n-k+1,\lambda})\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.13}
$$

Thus, by (2.13), we get

$$
Bel_n^{(\lambda)}(x,x,\cdots,x) = \sum_{k=0}^{n} x^k B_{n,k}((1)_{1,\lambda},(1)_{2,\lambda},\cdots,(1)_{n-k+1,\lambda}),\ (n \geq 0).
\tag{2.14}
$$

From (2.12) and (2.14), we have

$$
Bel_n^{(\lambda)}(\underbrace{x,x,\cdots,x}_{n-\text{times}}) = Bel_{n,\lambda}(x),\ (n \geq 0).
\tag{2.15}
$$

By (2.3), we get

$$
\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j (1)_{j,\lambda}\frac{t^j}{j!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda}\cdots x_{n-k+1}(1)_{n-k+1,\lambda})\frac{t^n}{n!}.
\tag{2.16}
$$

Thus, we note that

$$
\begin{aligned}
&\sum_{n=k}^{\infty} B_{n,k}((1)_{1,\lambda},(1)_{2,\lambda}\cdots(1)_{n-k+1,\lambda})\frac{t^n}{n!}\\
&= \frac{1}{k!}\left(\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right)^k = \frac{1}{k!}\left(\sum_{j=1}^{\infty}\binom{\frac{1}{\lambda}}{j}\lambda^j t^j\right)^k = \frac{1}{k!}((1+\lambda t)^{\frac{1}{\lambda}}-1)^k\\
&= \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!},\ (n,k \geq 0).
\end{aligned}
\tag{2.17}
$$

Comparing the coefficients on the both sides, we have

$$S_{2,\lambda}(n,k) = B_{n,k}((1)_{1,\lambda},(1)_{2,\lambda}\cdots(1)_{n-k+1,\lambda})$$
$$= \sum \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}\left(\frac{(1)_{1,\lambda}}{1!}\right)^{j_1}\left(\frac{(1)_{2,\lambda}}{2!}\right)^{j_2}\cdots\left(\frac{(1)_{n-k+1,\lambda}}{(n-k+1)!}\right)^{j_{n-k+1}}, \tag{2.18}$$

where $j_1+j_2+\cdots+j_{n-1+1}=k$, $j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n$ and $n,k\geq 0$ with $n\geq k$. From (2.17), we can derive the following equation:

$$\sum_{n=0}^{\infty} Bel_n^{(\lambda)}(x,x,\cdots,x)\frac{t^n}{n!} = \exp\left(x\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right)$$
$$= \exp\left(x\left(\sum_{j=0}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}-1\right)\right) = \frac{1}{e^x}\sum_{k=0}^{\infty}x^k\frac{1}{k!}\left(\sum_{j=0}^{\infty}(1)_{j,\lambda}\frac{t^j}{j!}\right)^k \tag{2.19}$$
$$= e^{-x}\sum_{k=0}^{\infty}x^k\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{e^x}\sum_{k=0}^{n}x^kS_{2,\lambda}(n,k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients on the both sides of (2.19), we have

$$Bel_n^{(\lambda)}(x,x,\cdots,x) = \frac{1}{e^x}\sum_{k=0}^{n}x^kS_{2,\lambda}(n,k)$$
$$= \frac{1}{e^x}\sum_{k=0}^{n}x^kB_{n,k}((1)_{1,\lambda},(1)_{2,\lambda},\cdots,(1)_{n-k+1,\lambda}), \tag{2.20}$$

and $B_{0,0}((1)_{1,\lambda},(1)_{2,\lambda},\cdots,(1)_{n-k+1,\lambda})=1$, for all $n,k\geq 0$.

3. Further remark

From (2.8), we easily note that

$$Bel_n^{(\lambda)}(x_1,x_2,\cdots,x_n) = \frac{d^n}{dt^n}\exp\left(\sum_{j=1}^{\infty}x_j(1)_{j,\lambda}\frac{t^j}{j!}\right)\Bigg|_{t=0},\ (n\geq 1), \tag{3.1}$$

and, by (2.16), we get

$$\begin{aligned}
&\sum_{n=k}^{\infty} B_{n,k}(0!,1!,2!,\cdots,(n-k)!)\frac{t^n}{n!} = \frac{1}{k!}(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots)^k \\
&= \frac{1}{k!}(-\log(1-k))^k = (-1)^k\sum_{n=k}^{\infty} S_1(n,k)(-1)^n\frac{t^n}{n!} \\
&= \sum_{n=k}^{\infty}(-1)^{n-k}S_1(n,k)\frac{t^n}{n!} = \sum_{n=k}^{\infty}|S_1(n,k)|\frac{t^n}{n!}.
\end{aligned} \tag{3.2}$$

Thus, by (3.2), we get

$$B_{n,k}(0!,1!,2!,\cdots,(n-k)!) = |S_1(n,k)|. \tag{3.3}$$

From (2.8), we note that

$$\exp\left(\sum_{j=1}^{\infty}(x)_j(1)_{j,\lambda}\frac{t^j}{j!}\right) = \sum_{n=0}^{\infty} Bel_n^{(\lambda)}(x_1,x_2,\cdots,x_n)\frac{t^n}{n!}. \tag{3.4}$$

By (3.4), we get

$$\begin{aligned}
&Bel_n^{(\lambda)}(x_1,x_2,\cdots,x_n) \\
&= \begin{vmatrix}
x_1(1)_{1,\lambda} & \binom{n-1}{1}x_2(1)_{2,\lambda} & \binom{n-1}{2}x_3(1)_{3,\lambda} & \binom{n-1}{3}x_4(1)_{4,\lambda} & \binom{n-1}{4}x_5(1)_{5,\lambda} & \cdots & x_n(1)_{n,\lambda} \\
-1 & x_1(1)_{1,\lambda} & \binom{n-2}{1}x_2(1)_{2,\lambda} & \binom{n-2}{2}x_3(1)_{3,\lambda} & \binom{n-2}{3}x_4(1)_{4,\lambda} & \cdots & x_{n-1}(1)_{n-1,\lambda} \\
0 & -1 & x_1(1)_{1,\lambda} & \binom{n-3}{1}x_2(1)_{2,\lambda} & \binom{n-3}{2}x_3(1)_{3,\lambda} & \cdots & x_{n-2}(1)_{n-2,\lambda} \\
0 & 0 & -1 & x_1(1)_{1,\lambda} & \binom{n-4}{1}x_2(1)_{2,\lambda} & \cdots & x_{n-3}(1)_{n-3,\lambda} \\
0 & 0 & 0 & -1 & x_1(1)_{1,\lambda} & \cdots & x_{n-4}(1)_{n-4,\lambda} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & x_1(1)_{1,\lambda}
\end{vmatrix}
\end{aligned} \tag{3.5}$$

From (2.15), we note that

$$
\begin{aligned}
&Bel_{n,\lambda}(x)\\
&=\begin{vmatrix}
x(1)_{1,\lambda} & x\binom{n-1}{1}(1)_{2,\lambda} & x\binom{n-1}{2}(1)_{3,\lambda} & x\binom{n-1}{3}(1)_{4,\lambda} & x\binom{n-1}{4}(1)_{5,\lambda} & \cdots & x(1)_{n,\lambda}\\
-1 & x(1)_{1,\lambda} & x\binom{n-2}{1}(1)_{2,\lambda} & x\binom{n-2}{2}(1)_{3,\lambda} & x\binom{n-1}{3}(1)_{4,\lambda} & \cdots & x(1)_{n-1,\lambda}\\
0 & -1 & x(1)_{1,\lambda} & x\binom{n-3}{1}(1)_{2,\lambda} & x\binom{n-3}{2}(1)_{3,\lambda} & \cdots & x(1)_{n-2,\lambda}\\
0 & 0 & -1 & x(1)_{1,\lambda} & x\binom{n-4}{1}(1)_{2,\lambda} & \cdots & x(1)_{n-3,\lambda}\\
0 & 0 & 0 & -1 & x(1)_{1,\lambda} & \cdots & x(1)_{n-4,\lambda}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -1 & x(1)_{1,\lambda}
\end{vmatrix}\\
&=x^{\binom{n+1}{2}}\begin{vmatrix}
(1)_{1,\lambda} & \binom{n-1}{1}(1)_{2,\lambda} & \binom{n-1}{2}(1)_{3,\lambda} & \binom{n-1}{3}(1)_{4,\lambda} & \binom{n-1}{4}(1)_{5,\lambda} & \cdots & (1)_{n,\lambda}\\
-1 & (1)_{1,\lambda} & \binom{n-2}{1}(1)_{2,\lambda} & \binom{n-2}{2}(1)_{3,\lambda} & \binom{n-2}{3}(1)_{4,\lambda} & \cdots & (1)_{n-1,\lambda}\\
0 & -1 & (1)_{1,\lambda} & \binom{n-3}{1}(1)_{2,\lambda} & \binom{n-3}{2}(1)_{3,\lambda} & \cdots & (1)_{n-2,\lambda}\\
0 & 0 & -1 & (1)_{1,\lambda} & \binom{n-4}{1}(1)_{2,\lambda} & \cdots & (1)_{n-3,\lambda}\\
0 & 0 & 0 & -1 & (1)_{1,\lambda} & \cdots & (1)_{n-4,\lambda}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & -1 & (1)_{1,\lambda}
\end{vmatrix}.
\end{aligned}
\tag{3.6}
$$

As is known, the λ- Stirling number of the first kind is defined as

$$
(x)_{n,\lambda}=\sum_{l=0}^{n}S_{1,\lambda}(n,l)x^{l},\ (n\geq 0),\quad \text{(see [8])}. \tag{3.7}
$$

Thus, by (3.7), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}Bel_{n}^{(\lambda)}(1,1,\cdots,1)\frac{t^{n}}{n!}=\exp\left(\sum_{j=1}^{\infty}(1)_{j,\lambda}\frac{t^{j}}{j!}\right)\\
&=\exp\left(\sum_{j=1}^{\infty}\left(\sum_{l=0}^{j}S_{1,\lambda}(j,l)\right)\frac{t^{j}}{j!}\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{\infty}\left(\sum_{l=0}^{j}S_{1,\lambda}(j,l)\right)\frac{t^{j}}{j!}\right)^{k}\\
&=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}B_{n,k}\left(\sum_{l=0}^{1}S_{1,\lambda}(1,\lambda),\sum_{l=0}^{2}S_{1,\lambda}(2,l),\cdots,\sum_{l=0}^{n-k+1}S_{1,\lambda}(n-k+1,l)\right)\frac{t^{n}}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}B_{n,k}\left(\sum_{l=0}^{1}S_{1,\lambda}(1,\lambda),\sum_{l=0}^{2}S_{1,\lambda}(2,l),\cdots,\sum_{l=0}^{n-k+1}S_{1,\lambda}(n-k+1,l)\right)\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{3.8}
$$

Comparing the coefficients on the both sides of (3.8), we have

$$Bel_n^{(\lambda)}(\underbrace{1,1,\cdots,1}_{n-\text{times}})$$
$$=\sum_{k=0}^{n}B_{n,k}\left(\sum_{l=0}^{1}S_{1,\lambda}(1,\lambda),\sum_{l=0}^{2}S_{1,\lambda}(2,l),\cdots,\sum_{l=0}^{n-k+1}S_{1,\lambda}(n-k+1,l)\right).$$

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Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

E-mail address: tkkim@kw.ac.kr